Topic – Problem based on 2nd order Diff. Equation with variable co-efficients- Method by changing the independent variable (contd.)

B.Sc III (Math Hons)

Date –

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Ex: → Solve

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

The Given Equation is

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$$

Here $P = \cot x$

$Q = 4 \operatorname{cosec}^2 x$

Now changing the independent variable x to z, the Given Equation ① is Transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y + X_1 = 0$$

Where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\left(\frac{dz}{dx}\right)}{\left(\frac{dz}{dx}\right)^2}$ $\quad Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$$Q_1 = \frac{4 \operatorname{cosec}^2 x}{\left(\frac{dz}{dx}\right)^2} = 4 \Rightarrow \frac{dz}{dx} = \operatorname{cosec} x$$

$$\Rightarrow \int dz = \int \operatorname{cosec} x \, dx$$

$$\Rightarrow z = \log \tan x/2$$

Also $\frac{d^2z}{dx^2} = -\operatorname{cosec} x \cdot \cot x$

$$\therefore P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x}{\operatorname{cosec}^2 x}$$

$$= 0$$

Hence the Given Equation reduces to

$$\frac{d^2y}{dz^2} + y = 0$$

or $(D^2 + 1)y = 0$

$\Rightarrow D^2 + 1 = 0$

$\Rightarrow D^2 = -1$

$\Rightarrow D = \pm i$

$\therefore y = C_1 \cos(z) + C_2$

$= C_1 \cos(2 \log \tan x/2) + C_2 \sin(2 \log \tan x/2)$

solve $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4$

Soln - Here $P = -1/x$

$Q = 4x^2$

$R = x^4$

Changing the independent variable from x to z, the equation reduces to the form

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{—— (1)}$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let us take $P_1 = 0$

$$\Rightarrow P\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0$$

$$-\frac{1}{x}\frac{dz}{dx} + \frac{d^2z}{dx^2} = 0$$

put $\frac{dz}{dx} = q$

then $-\frac{q}{x} + \frac{d}{dx}\left(\frac{dz}{dx}\right) = 0$

$$-\frac{q}{x} + \frac{dq}{dx} = 0$$

$$\frac{dq}{q} = \frac{dx}{x}$$

$\Rightarrow \log q = \log x$

$\Rightarrow q = x$

$\Rightarrow \frac{dz}{dx} = x$

$\Rightarrow \int dz = \int x\,dx \quad \Rightarrow z = \frac{x^2}{2}$

Now $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \dfrac{4x^2}{x^2} = 4$

$R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2} = \dfrac{x^4}{x^2} = x^2 = 2z.$

Hence from (1)

$$\frac{d^2y}{dz^2} + 4y = 2z \quad \text{——— (2)}$$

Auxiliary Equation from (2) is

$m^2 + 4 = 0 \Rightarrow m^2 = -4 \quad \therefore m = \pm 2i$

Hence $C.F = c_1 \cos 2z + c_2 \sin 2z$

$= c_1 \cos x^2 + c_2 \sin x^2$

The $PI = \dfrac{1}{D^2+4}(2z)$

$= \dfrac{1}{4\left(\frac{D^2}{4}+1\right)} 2z$

$= \dfrac{1}{4} \times 2z \left(1 + \dfrac{D^2}{4}\right)^{-1}$

$= \dfrac{1}{4} \cdot 2z \left[1 - \dfrac{D^2}{4} + \cdots\right]$

$= \dfrac{2z}{4} = \dfrac{x^2}{4} = .$

Hence the Comple Solution is

$y = c_1 \cos x^2 + c_2 \sin x^2 + x^2/4$

Solve

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0 \quad \text{--- (I)}$$

Here $P = \tan x$ $Q = \cos^2 x$ $R = 0$

changing the independent variable from x to z the equation reduces to the form

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{--- (II)}$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} \qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1 \Rightarrow \frac{\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = 1$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = \cos^2 x$$

$$\frac{dz}{dx} = \cos x$$

$$\int dz = \int \cos x \, dx$$

$$z = \sin x$$

$$\frac{dz}{dx} = \cos x$$

$$\frac{d^2z}{dx^2} = -\sin x$$

$$\therefore P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$$

$$= \frac{\tan x \cos x - \sin x}{\left(\frac{dz}{dx}\right)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos^2 x}{\cos^2 x} = 1$$

Hence from (11)

$$\frac{d^2y}{dz^2} + y = 0$$

$\therefore$ Auxiliary Equation

$$D^2 + 1 = 0$$

$$D^2 = -1 = \pm i$$

$\therefore$ Solution $= C_1 \cos z + C_2 \sin z$

$$= C_1 \cos(\sin x) + C_2 \sin(\sin x)$$